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AN ALGEBRA OF CONTINUED FRACTIONS

being

A Thesis Presented to the Graduate Faculty
of the Fort Hays Kansas State College in
Partial Fulfillment of the Requirements for
the Degree of Master of Arts

by

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August 1, 1961

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AN ALGEBRA OF CONTINUED FRACTIONS

(An Abstract)

by
Kay W. Dundas

The purpose of this paper is to develop an algebraic system for continued fractions which represent the positive square roots of natural numbers. These continued fractions are shown to have the general form;

$$\sqrt{N} = a + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \dots$$

where a and b are natural numbers. It is also shown that any continued fraction having this form represents the positive square root of some natural number.

Equality of continued fractions having this particular form is defined and is shown to be a true equivalence relationship. The operations of addition and multiplication are defined and it is shown that these operations follow the associative, commutative, and distributive laws.

In chapter three a method for obtaining simple continued fractions from the general form is developed. This method is applicable when the general form represents the square root of a natural number which is not a perfect square.

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INTRODUCTION

The continued fraction is an interesting expression which is encountered in elementary courses in the Theory of Numbers. The terminating continued fraction represents a rational number and does not require any new concepts of the student. The infinite continued fraction, however, introduces several new concepts and ideas to most students. Unlike the decimal representations which have no pattern, the continued fraction representing an irrational number often follows a definite pattern. This fact makes it possible to devise an algebraic system of operations for continued fractions. The first problem of this thesis is to develop such a system.

Continued fraction representations for many different irrational numbers are known and it would be beyond the scope of one paper to incorporate all such representations in one system. Consequently this paper will be limited to a treatment of one type of irrational number, those represented by the positive square roots of the natural numbers (1,2,3, . . .). This special type of irrational number is represented by a definite form of infinite continued fraction and special notation is introduced to aid in the development of the algebra of such numbers.

When working with continued fractions it is often necessary to obtain the simple continued fraction which represents some real number. The conventional methods for doing this are often tedious and time consuming. The second problem of this thesis is to develop an efficient method for obtaining the simple continued fraction of the positive square root of a natural number which is not a perfect square.

The purpose of chapter one of this paper is to acquaint the reader with the terminology and general subject matter of continued fractions. The second chapter contains the development of an algebra of infinite continued fractions and chapter three consists of a method for obtaining simple continued fractions.

CHAPTER I

CONTINUED FRACTIONS

Definition: An expression F of the form,

$$F = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}} \quad (1-1)$$

where the a 's and b 's are integers is called a continued fraction. Such an expression can be written:

$$F = a_0 + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_3}{a_3} + \dots \quad (1-2)$$

with all of the (+) signs, except the first, placed between the a_n terms instead of being in the position indicating addition of rational numbers. [(1) p.387]

The quantities $a_0, \frac{b_1}{a_1}, \frac{b_2}{a_2}, \dots, \frac{b_n}{a_n}$, are called elements of F . If the number of elements is finite, F is called a terminating continued fraction. If the elements are determined by a rule such that there is no last element then F is called a non-terminating or infinite continued fraction. Non-terminating continued fractions are sometimes written:

$$F = a_0 + \prod_{n=1}^{\infty} \frac{b_n}{a_n} \quad [(5) p. 105]$$

The terminating continued fraction obtained by stopping at any particular element of a given continued fraction is called a convergent. Thus the zero-th, first, second, . . . , convergents of the general continued fraction are: (1-1)

$$a_0, a_0 + \frac{b_1}{a_1}, a_0 + \frac{b_1}{a_1} + \frac{b_2}{a_2}, \dots$$

If the n th convergent is denoted by $\frac{A_n}{B_n}$ then:

$$\frac{A_0}{B_0} = \frac{a_0}{1}; \quad \frac{A_1}{B_1} = a_0 + \frac{b_1}{a_1} = \frac{a_0 a_1 + b_1}{a_1};$$

$$\text{and } A_0 = a_0, \quad A_1 = a_0 a_1 + b_1, \quad B_0 = 1, \text{ and } B_1 = a_1.$$

$$\frac{A_2}{B_2} \text{ may be obtained from } \frac{A_1}{B_1} \text{ by substituting } a_1 + \frac{b_2}{a_2}$$

for a_1 . Thus,

$$\frac{A_2}{B_2} = \frac{a_0(a_1 + b_2/a_2) + b_1}{(a_1 + b_2/a_2)} = \frac{a_0(a_2 a_1 + b_2) + b_1 a_2}{(a_1 a_2 + b_2)}$$

$$= \frac{a_2(a_0 a_1 + b_1) + b_2 a_0}{(a_2 a_1 + b_2)} = \frac{a_2 A_1 + b_2 A_0}{a_2 B_1 + b_2 B_0}$$

$$\text{and } A_2 = a_2 A_1 + b_2 A_0, \quad B_2 = a_2 B_1 + b_2 B_0.$$

Proceeding thus it can be shown that A_n and B_n are defined for successive values of n by the equations:

$$A_n = a_n A_{n-1} + b_n A_{n-2}, \quad B_n = a_n B_{n-1} + b_n B_{n-2} \quad (1-3)$$

These are called recursion formulas. If A_{-1} and B_{-1} , which occur in the right hand members are defined as $A_{-1} = 1$ and $B_{-1} = 0$ the equations will hold for all n greater than or equal to one. [(1) p. 389]

For a terminating continued fraction F ; $F = \frac{A_n}{B_n}$, where $\frac{A_n}{B_n}$ is the last convergent, but if F is a non-terminating continued fraction, then F must be evaluated as a limit;

$$F = \lim_{n \rightarrow \infty} \frac{A_n}{B_n}.$$

Definition: An expression of the form;

$$a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \dots \quad (1-4)$$

where a_1 is a positive integer (a_0 may be zero) is called a simple continued fraction. The a_0, a_1, a_2, \dots are called partial quotients. If x_n is defined by;

$$x_0 = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots$$

$$x_1 = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \dots$$

$$x_2 = a_2 + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \dots$$

then x_0, x_1, x_2, \dots are called complete quotients. Therefore the continued fraction F ;

$$F \equiv x_0 = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_{n-1}} + \frac{1}{x_n} \quad (1-5)$$

is obtained from its n th convergent by substituting the complete quotient x_n for the partial quotient a_n .

Any rational number can be expressed as a terminating simple continued fraction. To obtain such a representation the rational number is expressed first as a mixed number with the integral part added to the proper fraction. For example;

$$\frac{19}{5} = 3 \frac{4}{5} = 3 + \frac{4}{5}$$

The proper fraction is then written as the quotient of 1 and the reciprocal of this proper fraction;

$$\frac{19}{5} = 3 + \frac{1}{\frac{5}{4}} \cdot$$

Now since the inverted fraction is greater than one the process can be repeated until a continued fraction of the desired form is obtained.

$$\frac{19}{5} = 3 + \frac{1}{1} + \frac{1}{4} \cdot$$

Any terminating simple continued fraction may be converted into a rational number by essentially the same process with the steps reversed. For example;

$$1 + \frac{1}{3} + \frac{1}{1} + \frac{1}{4} = 1 + \frac{1}{3} + \frac{1}{\frac{5}{4}} =$$

$$1 + \frac{1}{3} + \frac{4}{5} = 1 + \frac{1}{\frac{5}{19}} = 1 + \frac{5}{19} = \frac{24}{19}$$

More generally if;

$$F_n = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$$

then for $n = 1$,

$$F_1 = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}$$

for $n = 2$,

$$F_2 = a_0 + \frac{1}{a_1} + \frac{1}{a_2} = \frac{a_0 a_1 a_2 + a_0 + a_2}{a_1 a_2 + 1}$$

for $n = 3$,

$$F_3 = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} = \frac{a_0 a_1 a_2 a_3 + a_0 a_1 + a_0 a_3 + a_2 a_3 + 1}{a_1 a_2 a_3 + a_1 + a_3}$$

The numerator and denominator follow a definite pattern

where the numerator is some function of (a_0, a_1, \dots, a_n)

and the denominator is the same function of (a_1, a_2, \dots, a_n) .

The first person to calculate the value of this function was

Euler. He contrived the following rule for its evaluation.

EULER'S RULE: First take the product of all the terms. Then take every product that can be obtained by omitting any pair of consecutive terms. Then take every product that can be obtained by omitting any two separate pairs of consecutive terms, and so on.

The sum of all such products gives the value of (a_0, a_1, \dots, a_n) .

The product obtained by omitting all the terms is given the value 1 to fit the pattern correctly. [(2) p. 83]

The n th convergent in this notation becomes;

$$\frac{A_n}{B_n} = \frac{(a_0, a_1, a_2, \dots, a_n)}{(a_1, a_2, a_3, \dots, a_n)}$$

The recursion formulas for simple continued fractions have the form;

$$A_n = a_n A_{n-1} + A_{n-2} \quad , \quad B_n = a_n B_{n-1} + B_{n-2}$$

or in different symbols;

$$(a_0, a_1, \dots, a_n) = a_n(a_0, a_1, \dots, a_{n-1}) + (a_0, a_1, \dots, a_{n-2})$$

$$(a_1, a_2, \dots, a_n) = a_n(a_1, a_2, \dots, a_{n-1}) + (a_1, a_2, \dots, a_{n-2})$$

These recursion formulas can be developed directly from the definition of (a_0, a_1, \dots, a_n) and Euler's rule for its evaluation. [(2) p. 85]

A simple relation that is satisfied by any two consecutive convergents exists. This relationship which is of the greatest importance may be stated in the following form;

$$A_n B_{n-1} - B_n A_{n-1} = (-1)^{n-1} \quad (1-6)$$

For example, if $n = 1$,

$$A_1 B_0 - B_1 A_0 = (a_0 a_1 + 1) - a_0 a_1 = 1$$

This relation can be proved generally by use of the recursion formulas. If the values of A_n and B_n are substituted from the formulas;

$$\begin{aligned} A_n B_{n-1} - B_n A_{n-1} &= (a_n A_{n-1} + A_{n-2}) B_{n-1} - (a_n B_{n-1} + B_{n-2}) A_{n-1} \\ &= -(A_{n-1} B_{n-2} - B_{n-1} A_{n-2}) \end{aligned}$$

Thus the expression on the left, call it Δ_n , has the property that $\Delta_n = -\Delta_{n-1}$. Since $\Delta_1 = 1$, $\Delta_2 = -1$, in general $\Delta_n = (-1)^{n-1}$.

$$\text{Therefore; } A_n B_{n-1} - B_n A_{n-1} = (-1)^{n-1}$$

One immediate consequence of this relation is that A_n and B_n are always relatively prime, since any common factor would have to be a factor of 1. Thus the fraction $\frac{A_n}{B_n}$ is always in lowest terms.

If the relation is written in a different form;

$$\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = \frac{(-1)^{n-1}}{B_{n-1}B_n}$$

the fact that the convergents are alternately less than, and greater than the final value of the fraction is evident since the difference of consecutive convergents is positive if n is odd and negative if n is even. Also, since the numbers B_0, B_1, \dots, B_n increase steadily, the difference decreases steadily as n increases.

This fact can be shown geometrically for a specific example.

$$\frac{63}{50} = 1 + \frac{1}{3} + \frac{1}{1} + \frac{1}{5} + \frac{1}{1} + \frac{1}{1}$$

$$\frac{A_0}{B_0} = 1 = 1.000$$

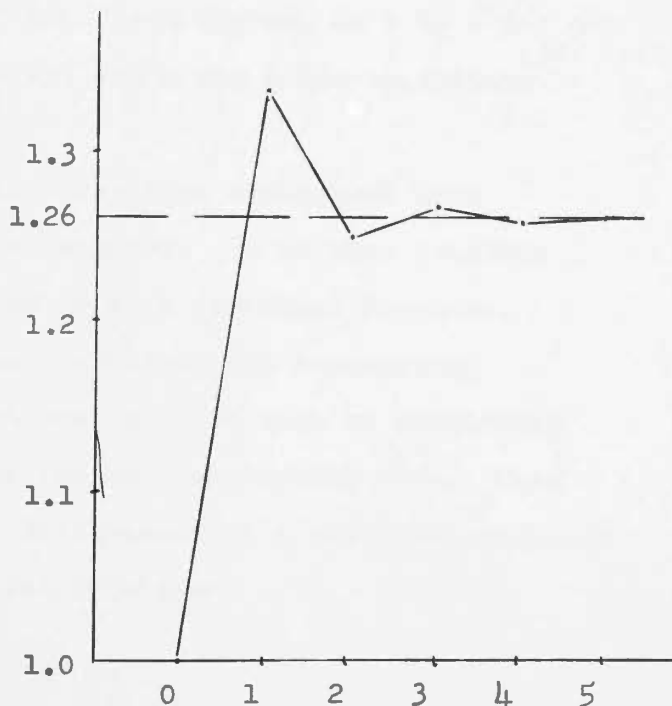
$$\frac{A_1}{B_1} = \frac{4}{3} \approx 1.333$$

$$\frac{A_2}{B_2} = \frac{5}{4} = 1.250$$

$$\frac{A_3}{B_3} = \frac{29}{23} \approx 1.261$$

$$\frac{A_4}{B_4} = \frac{34}{27} \approx 1.259$$

$$\frac{A_5}{B_5} = \frac{63}{50} = 1.260$$



This relationship is also useful in solving the equation $ax + by = 1$ for integral values of x and y . This equation will, of course, have solutions only if a and b are relatively prime. To solve this equation the fraction $\frac{a}{b}$ is written as a continued fraction where;

$$\frac{A_n}{B_n} = \frac{a}{b} \quad \text{and} \quad \frac{A_{n-1}}{B_{n-1}} \quad \text{is the convergent preceding}$$

the n th convergent. Then;

$$A_n B_{n-1} - B_n A_{n-1} = (-1)^{n-1} \quad \text{or} \quad a B_{n-1} - b A_{n-1} = (-1)^{n-1}$$

Thus either $(B_{n-1}, -A_{n-1})$ or $(-B_{n-1}, A_{n-1})$ is a solution of the original equation depending on the value of n . If n is even the first is a solution, if n is odd the latter is a solution. This method can also be applied to solving indeterminate equations of the first degree, $ax + by = c$ where a , b , and c are integers and a and b are relatively prime.

So far, all continued fractions considered have been expressions of rational numbers. It is also possible to express an irrational number as a continued fraction. Common sense would tell one that since all terminating continued fractions are rational numbers then an irrational number must be expressed in the non-terminating form. This is shown to be true in the development of a continued fraction from an irrational number which follows.

Let p be any irrational number and $a_0 = [p]$.* Then;

$$p = a_0 + p' \quad \text{where } 0 < p' < 1$$

Let

$$p' = \frac{1}{p_1} \quad \text{where } p_1 > 1$$

then

$$p = a_0 + \frac{1}{p_1}.$$

But p_1 is irrational so the operation can be repeated,

$$p = a_0 + \frac{1}{a_1 + \frac{1}{p_2}} \quad \text{where } a_1 = [p_1]$$

$$\text{and } p_2 > 1$$

By continuing the operation $n+1$ times the expression,

$$p = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{1}{p_{n+1}}$$

is obtained where $a_i = [p_i]$ for $i = 1, 2, 3, \dots$ and p_{n+1} is irrational and greater than one.

From Euler's rule and the recursion formulas it can now be shown that this expression does converge to p as n increases without bound.

$$p = \frac{(a_0, a_1, \dots, a_n, p_{n+1})}{(a_1, a_2, \dots, a_n, p_{n+1})} = \frac{p_{n+1}A_n + A_{n-1}}{p_{n+1}B_n + B_{n-1}}$$

$$\text{and; } \left| p - \frac{A_n}{B_n} \right| = \left| \frac{p_{n+1}A_n + A_{n-1}}{p_{n+1}B_n + B_{n-1}} - \frac{A_n}{B_n} \right| = \left| \frac{A_{n-1}B_n - B_{n-1}A_n}{B_n(p_{n+1}B_n + B_{n-1})} \right|$$

*The notation $[p]$, where p is any real number, will be defined throughout this paper as: $[p]$ equals the greatest integer less than or equal to p .

$$= \left| \frac{+1}{B_n(p_{n+1}B_n + B_{n-1})} \right| < \frac{1}{B_n B_{n-1}}$$

or

$$\left| p - \frac{A_n}{B_n} \right| < \frac{1}{B_n B_{n-1}}$$

B_1, B_2, B_3, \dots are strictly increasing natural numbers so as n increases without bound so does B_n .

Therefore;

$$\lim_{n \rightarrow \infty} \left| p - \frac{A_n}{B_n} \right| \leq \lim_{n \rightarrow \infty} \frac{1}{B_n B_{n-1}} = 0$$

$$\lim_{n \rightarrow \infty} \left| p - \frac{A_n}{B_n} \right| = 0$$

or $\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = p$

Therefore;

$$a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \dots, + \frac{1}{a_n} + \dots$$

converges to the value of the original number p as n increases without limit and therefore is a true representation of p .

The representation of an irrational number by an infinite continued fraction suggests another question. If any infinite sequence of numbers (a_0, a_1, a_2, \dots) all of which are natural numbers, except possibly the first which may be zero, is used in forming the simple continued fraction;

$$a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \dots$$

can meaning be attached to the expression and will it be irrational? The fact that this continued fraction has a limit is most easily proved by considering the even convergents, $\frac{A_0}{B_0}, \frac{A_2}{B_2}, \dots$ and the odd convergents, $\frac{A_1}{B_1}, \frac{A_3}{B_3}, \dots$

The even convergents form an increasing sequence which is bounded above, since all are less than $\frac{A_1}{B_1}$. The odd convergents form a decreasing sequence which is bounded below, since all are greater than $\frac{A_0}{B_0}$. Now using the relation (1-6)

$$\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = \frac{+1}{B_n B_{n-1}}$$

$$\lim_{n \rightarrow \infty} \left[\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right] = \lim_{n \rightarrow \infty} \frac{+1}{B_n B_{n-1}} = 0$$

$$\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = \lim_{n \rightarrow \infty} \frac{A_{n-1}}{B_{n-1}}$$

Without loss of generality, n can be assumed to be even therefore making $n-1$ odd. Consequently both even and odd convergents approach the same limit as n increases without bound. If this limit is denoted as p then it can be shown that;

$$a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \dots$$

is the continued fraction that would be obtained by the method previously mentioned. [(2) p. 93]

As indicated on page three of this paper, an infinite continued fraction is adequately determined only when some rule is stated so that the n th term can be obtained from the $(n-1)$ st term. Probably the best known and widely studied rule for forming continued fractions is that of the recurring period. If a single partial quotient or a group of consecutive partial quotients recur continually in a continued fraction it is called a recurring continued fraction. If the recurring period of the fraction is denoted as;

$$a_n + \frac{1}{a_{n+1} + \frac{1}{a_{n+1} + \dots + \frac{1}{a_{n+k}}}}$$

then the recurring portion of the continued fraction may be evaluated by setting;

$$x = a_n + \frac{1}{a_{n+1} + \frac{1}{a_{n+2} + \dots + \frac{1}{a_{n+k} + \frac{1}{x}}}}$$

and solving for x . This produces a quadratic whose positive root is the value of the recurring portion. [(3) p. 573]

The general theorem that any quadratic irrational number has a continued fraction representation which is periodic after a certain stage was first proved by Lagrange in 1770. [(2) p. 97]

The recurring period of the continued fraction is called the cycle while the non-recurring portion, if such exists, is called the acyclic part. A periodic continued fraction with no acyclic part is said to be purely periodic.

The purely periodic continued fraction represents a particular kind of quadratic irrational number.

If,

$$p = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{1}{p}$$

then,

$$p = \frac{pA_n + A_{n-1}}{pB_n + B_{n-1}}$$

If q is defined to be the continued fraction obtained by reversing the period of p , that is

$$q = a_n + \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}} + \dots + \frac{1}{a_0} + \frac{1}{q}$$

then,

$$q = \frac{qA_n + B_n}{qA_{n-1} + B_{n-1}}$$

This result in which the $A_n, A_{n-1}, B_n, B_{n-1}$, are the same in both equations follows from the fact, based on Euler's rule, that the value of (a_0, a_1, \dots, a_n) is unchanged if the terms are taken in the opposite order. Solving the first equation for p and the second for $-\frac{1}{q}$.

$$p = \frac{A_n - B_{n-1} + \sqrt{(B_{n-1} - A_n)^2 + 4B_n A_{n-1}}}{2B_n}$$

$$-\frac{1}{q} = \frac{A_n - B_{n-1} - \sqrt{(B_{n-1} - A_n)^2 + 4B_n A_{n-1}}}{2B_n}$$

From this it follows that $-\frac{1}{q}$ is the algebraic conjugate of p . Since q is greater than one, $-\frac{1}{q}$ lies between -1 and 0 .

Hence any purely periodic continued fraction represents a quadratic irrational number p which is greater than 1, and whose conjugate $-\frac{1}{q}$ lies between -1 and 0. q is defined by the continued fraction which is obtained by reversing the period of the continued fraction which represents p . The converse of this rule was first proved by Galois in 1828, though the result was implicit in the earlier work of Lagrange. [(2) p. 100]

With this background it is a simple matter to develop the form for \sqrt{N} where N is a natural number but not a perfect square. The conjugate of \sqrt{N} is $-\sqrt{N}$ which does not lie between -1 and 0. Hence the continued fraction for \sqrt{N} cannot be purely periodic. However the number $a_0 + \sqrt{N}$, where $a_0 \equiv [\sqrt{N}]$, has the conjugate $a_0 - \sqrt{N}$ which does lie between -1 and 0.

Hence the continued fraction;

$$p = a_0 + \sqrt{N} = 2a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{1}{2a_0} + \dots$$

and

$$q = a_n + \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}} + \dots + \frac{1}{a_1} + \frac{1}{2a_0} + \frac{1}{a_n} + \dots$$

But $-\frac{1}{q} = a_0 - \sqrt{N}$ which implies that $q = \frac{1}{\sqrt{N} - a_0}$.

This form may be evaluated by noting that,

$$\sqrt{N} - a_0 = (a_0 + \sqrt{N}) - 2a_0 \text{ from which,}$$

$$\sqrt{N} - a_0 = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} + \frac{1}{2a_0} + \dots$$

Therefore two expressions are obtained for q ;

$$q = \frac{1}{\sqrt{N} - a_0} = a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \dots + \frac{1}{a_n} + \frac{1}{2a_0} + \frac{1}{a_1} + \dots$$

and

$$q = a_n + \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}} + \dots + \frac{1}{a_1} + \frac{1}{2a_0} + \frac{1}{a_n} + \dots$$

Therefore the two expressions are equal and the respective partial quotients must be equal, $a_1 = a_n$, $a_2 = a_{n-1}$, $a_3 = a_{n-2}$, and so forth. Hence the continued fraction for N is necessarily of the form;

$$\sqrt{N} = a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_2} + \frac{1}{a_1} + \frac{1}{2a_0} + \dots$$

The period begins immediately after the first term a_0 and it consists of the symmetrical partial quotients $a_1, a_2, \dots, a_2, a_1$, followed by $2a_0$. [(2) p. 104]

The continued fraction representation of \sqrt{N} is treated in greater detail in chapters 2 and 3.

There are not many irrational numbers, other than quadratic irrationals, whose continued fractions are known to have any features of regularity. For instance, it is not known whether the terms of the continued fraction for $\sqrt[3]{2}$, which begins;

$$\sqrt[3]{2} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{1} + \frac{1}{1} + \frac{1}{4} + \frac{1}{1} + \dots$$

form any recognizable pattern or not; and no method by which such a problem can be attacked is known. [(2) p. 107]

Euler, who invented the symbol e and calculated its value to 23 places, found several interesting continued fraction representations of functions involving the irrational number e . Some of the more important are;

$$e = 2 + \frac{1}{1} + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots \quad [(4) \text{ p. } 85]$$

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6}}}}}}}} \dots \quad [(2) \text{ p. } 107]$$

where 2, 4, 6, . . . are separated by two ones each time.

$$\sqrt{e} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{1 + \frac{1}{1 + \frac{1}{9}}}}}} \dots \quad [(4) \text{ p. } 86]$$

$$\frac{e-1}{e+1} = \frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \frac{1}{14} + \dots + \frac{1}{4n-2} + \dots$$

more generally,

$$\frac{\frac{2}{e^k} - 1}{\frac{2}{e^k} + 1} = \frac{1}{k} + \frac{1}{3k} + \frac{1}{5k} + \dots + \frac{1}{(2n-1)k} + \dots \quad [(2) \text{ p. } 107]$$

Other important irrational numbers for which continued fraction representations are known include π and some logarithms. The following continued fraction was communicated by Lord Brouncker (1620-1684) to John Wallis.

$$\pi = \frac{4}{1} + \frac{1^2}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \frac{7^2}{2} + \dots \quad [(4) \text{ p. } 78]$$

The continued fraction for the natural logarithm of two has the form;

$$\log 2 = \frac{1}{1} + \frac{1^2}{1} + \frac{2^2}{1} + \frac{3^2}{1} + \dots + \frac{(n-1)^2}{1} + \dots \quad [(1) \text{ p. } 396]$$

CHAPTER II

AN ALGEBRA OF CONTINUED FRACTIONS

This chapter is devoted to the development of an algebra of those continued fractions used to represent the \sqrt{N} , where N is any natural number. When \sqrt{N} is an irrational number, it was shown in the previous chapter to have a unique form when expressed as a simple continued fraction. The \sqrt{N} can be expressed in another special form as a general continued fraction. This form is;

$$\sqrt{N} = a + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \dots$$

The following theorems show that the \sqrt{N} has this form and that any expression of this form represents the positive square root of some natural number.

Theorem 1 : If N is a natural number then \sqrt{N} can be expressed in the form;

$$\sqrt{N} = a + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \dots$$

where a takes any one of the values $1, 2, 3, \dots$ and b takes any one of the values $0, 1, 2, \dots$

Consider the equation;

$$x^2 - 2[\sqrt{N}]x - N + [\sqrt{N}]^2 = 0$$

When solved by the quadratic formula this equation yields

the solution, $x = [\sqrt{N}] \pm \sqrt{N}$. However, if it is solved in the following manner the solution takes a different form.

$$x^2 = 2[\sqrt{N}]x + N - [\sqrt{N}]^2$$

dividing through by x gives,

$$x = 2[\sqrt{N}] + \frac{N - [\sqrt{N}]^2}{x}.$$

If the x in the denominator is replaced by the entire right hand side of the equation;

$$x = 2[\sqrt{N}] + \frac{N - [\sqrt{N}]^2}{2[\sqrt{N}] + \frac{N - [\sqrt{N}]^2}{x}}.$$

If this process is continued indefinitely x takes the form;

$$x = 2[\sqrt{N}] + \frac{N - [\sqrt{N}]^2}{2[\sqrt{N}] + \frac{N - [\sqrt{N}]^2}{2[\sqrt{N}] + \frac{N - [\sqrt{N}]^2}{2[\sqrt{N}] + \dots}}.$$

Now if the two positive solutions are equated;

$$[\sqrt{N}] + \sqrt{N} = 2[\sqrt{N}] + \frac{N - [\sqrt{N}]^2}{2[\sqrt{N}] + \frac{N - [\sqrt{N}]^2}{2[\sqrt{N}] + \dots}}$$

or

$$\sqrt{N} = [\sqrt{N}] + \frac{N - [\sqrt{N}]^2}{2[\sqrt{N}] + \frac{N - [\sqrt{N}]^2}{2[\sqrt{N}] + \dots}}$$

Therefore \sqrt{N} is of the form;

$$\sqrt{N} = a + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \dots$$

where $a = [\sqrt{N}]$ and $b = N - [\sqrt{N}]^2$.

Theorem 2 : If a and b are natural numbers then;

$$a + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \dots$$

is equal to \sqrt{N} where N is also a natural number.

Let

$$x = a + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \dots$$

then

$$x = a + \frac{b}{a + a} + \frac{b}{2a} + \frac{b}{2a} + \dots$$

$$= a + \frac{b}{a + x}$$

$$x - a = \frac{b}{a + x}$$

$$(x - a)(x + a) = b$$

$$x^2 - a^2 = b$$

$$x^2 = a^2 + b$$

$$x = \sqrt{a^2 + b}$$

Therefore;

$$\sqrt{N} = a + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \dots$$

where $N = a^2 + b$. This is a natural number since a and b are natural numbers.

This form is used exclusively in the remainder of this chapter so for convenience the special notation

$(a,b) \equiv a + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \dots$ is adopted.

With this notation the results of the two previous theorems can be stated as;

Theorem 1: $\sqrt{N} = ([\sqrt{N}], N - [\sqrt{N}]^2)$

Theorem 2: $(a,b) = \sqrt{a^2 + b}$.

With the form of the continued fraction for \sqrt{N} established and a convenient notation agreed upon the development of an algebra may be started by defining equality with the following theorem.

Theorem 3 : (a,b) is equal to (c,d) if and only if $a^2 + b$ equals $c^2 + d$.

From theorem 2; $(a,b) = \sqrt{a^2 + b}$ and $(c,d) = \sqrt{c^2 + d}$.

If;

$$a^2 + b = c^2 + d \quad \text{then} \quad \sqrt{a^2 + b} = \sqrt{c^2 + d}$$

from which it follows that $(a,b) = (c,d)$.

Conversely, if $(a,b) = (c,d)$ then;

$$\sqrt{a^2 + b} = \sqrt{c^2 + d}$$

$$[\sqrt{a^2 + b}]^2 = [\sqrt{c^2 + d}]^2$$

$$a^2 + b = c^2 + d$$

which completes the proof of the theorem.

This definition of equality is shown to be a true equivalence relationship by the following corollaries.

Corollary 1 : Equality is reflexive, that is $(a,b) = (a,b)$.

$$(a,b) = (a,b) \text{ if } a^2 + b = a^2 + b$$

which is true therefore; $(a,b) = (a,b)$

Corollary 2 : Equality is symmetrical. If $(a,b) = (c,d)$

then $(c,d) = (a,b)$.

$$\text{If; } (a,b) = (c,d) \text{ then } a^2 + b = c^2 + d$$

and

$$c^2 + d = a^2 + b, \text{ therefore } (c,d) = (a,b).$$

Corollary 3 : Equality is transitive. If $(a,b) = (c,d)$ and

$(c,d) = (e,f)$ then $(a,b) = (e,f)$.

$$(a,b) = (c,d) \text{ implies that } a^2 + b = c^2 + d$$

$$(c,d) = (e,f) \text{ implies that } c^2 + d = e^2 + f$$

but the transitive law is valid for equality of natural numbers so;

$$a^2 + b = e^2 + f, \text{ and therefore } (a,b) = (e,f).$$

Corollary 4 : Equality is determinative. For any (a,b)

and (c,d) , either $(a,b) = (c,d)$ or $(a,b) \neq (c,d)$.

Certainly for any natural numbers a, b, c , and d either $a^2 + b = c^2 + d$ or $a^2 + b \neq c^2 + d$. Because the equality relationship, $(a,b) = (c,d)$ if and only if $a^2 + b = c^2 + d$ is symmetric for any (a,b) and (c,d) if;

$$a^2 + b = c^2 + d \quad \text{then } (a,b) = (c,d)$$

and if;

$$a^2 + b \neq c^2 + d \quad \text{then } (a,b) \neq (c,d)$$

Upon examining the (a,b) expression and the equality relations several facts can be discovered about the nature of this expression. Two of the more apparent facts are (1) that the expression is not always an irrational number and (2) that like common fractions, several continued fractions may have the same value. The following theorems deal with these facts more explicitly.

Theorem 4 : The expression (a,b) is a natural number if b is zero or if $b = 2an + n^2$, where n is a natural number.

$$\text{If } b = 0$$

$$(a,b) = (a,0) = \sqrt{a^2 + 0} = a$$

$$\text{If } b = 2an + n^2$$

$$(a,b) = (a, 2an + n^2) = \sqrt{a^2 + 2an + n^2}$$

$$= \sqrt{(a+n)^2} = a + n$$

Corollary 1 : $(a-n, 2an - n^2) = (a,0)$

$$b = 2an - n^2 = 2an - 2n^2 + n^2 = 2(a-n)n + n^2$$

therefore;

$$(a-n, 2an - n^2) \text{ is rational.}$$

$$(a-n, 2an-n^2) = \sqrt{(a-n)^2 + 2an - n^2} = \sqrt{a^2 - 2an + n^2 + 2an - n^2}$$

$$= \sqrt{a^2} = a = (a,0)$$

Corollary 2 : $1 = \frac{k+1}{k} + \frac{k+1}{k} + \frac{k+1}{k} + \dots$

where k is any natural number.

From the theorem, $a + n = (a, 2an + n^2)$

If, $n = 1$ then $a + 1 = (a, 2a + 1) =$

$$= a + \frac{2a+1}{2a} + \frac{2a+1}{2a} + \frac{2a+1}{2a} + \dots$$

Substituting $\frac{k}{2}$ for a gives;

$$\frac{k}{2} + 1 = \frac{k}{2} + \frac{k+1}{k} + \frac{k+1}{k} + \frac{k+1}{k} + \dots$$

or

$$1 = \frac{k+1}{k} + \frac{k+1}{k} + \frac{k+1}{k} + \dots$$

Definition : The continued fraction (a, b) which represents

\sqrt{N} is in "lowest terms" when $a = [\sqrt{N}]$.

Theorem 5 : $(a, b) = \sqrt{N}$ will be in lowest terms only if b is less than or equal to $2a$.

Assume $b > 2a$ and $a = [\sqrt{N}]$, then $b = 2a + k$ where $k \geq 1$
 and $(a, b) = (a, 2a + k) = (a, 2a + 1 + k - 1) = \sqrt{a^2 + 2a + 1 + k - 1}$
 $= \sqrt{(a + 1)^2 + k - 1}$

then;

$$(a + 1)^2 + k - 1 = N \quad \text{where } k - 1 \geq 0$$

therefore;

$$(a + 1)^2 \leq N \quad \text{or} \quad a + 1 \leq \sqrt{N}$$

Therefore, $a + 1$ is included in \sqrt{N} so $a \neq [\sqrt{N}]$ which contradicts the original assumption. Hence a must equal $[\sqrt{N}]$ only if b is less than or equal to $2a$.

The following theorem can be used to reduce (a,b) to its lowest terms.

Theorem 6 : (a,b) equals $(a + n, b - 2an - n^2)$ where n is an integer.

$$(a,b) = \sqrt{a^2 + b} = \sqrt{a^2 + 2an + n^2 + b - 2an - n^2} = \sqrt{(a + n)^2 + b - 2an - n^2} = (a + n, b - 2an - n^2)$$

The result of this theorem shows that the reducing process is one of increasing the a term and decreasing the b term. To apply the theorem n must be chosen so that the final form will be in lowest terms. The b term of the result must therefore be less than or equal to twice the a term, $(b - 2an - n^2) \leq 2(a + n)$. This will be accomplished if n is large enough, but if n is chosen too large the b term will become negative. Since (a,b) is not defined for negative integers n must be chosen so that; $0 \leq (b - 2an - n^2)$. Solving this inequation for n ;

$$0 \leq b - 2an - n^2$$

$$n^2 + 2an \leq b$$

$$n^2 + 2an + a^2 \leq a^2 + b$$

$$(n + a)^2 \leq a^2 + b$$

$$n + a \leq \sqrt{a^2 + b}$$

$$n \leq \sqrt{a^2 + b} - a$$

To reduce to lowest terms n must be chosen as large as possible and n must be an integer. Therefore the proper

value would be the greatest integer less than or equal to $\sqrt{a^2 + b} - a$ or $\lceil \sqrt{a^2 + b} - a \rceil$. Since a is an integer this becomes $n = \lceil \sqrt{a^2 + b} \rceil - a$. This result could be obtained intuitively from the definition of lowest terms which states that if $(a,b) = \sqrt{N}$, a must equal $\lceil \sqrt{N} \rceil$ for (a,b) to be in lowest terms. If a is less than $\lceil \sqrt{N} \rceil$ then one must add enough to a to obtain the value of $\lceil \sqrt{N} \rceil$. If n is added to a then;

$$a + n = \lceil \sqrt{N} \rceil \quad \text{or} \quad n = \lceil \sqrt{N} \rceil - a$$

which is the same value of n found above since $N = a^2 + b$.

Two operations are defined for this system. They are called addition and multiplication and are defined in terms of the conventional operations with the same names. When the operation for this system is intended the symbol \oplus is used for addition and \otimes is used for multiplication. When these symbols occur without the circle the conventional operation is intended.

Definition : The addition \oplus of the square roots of two natural numbers M and N is defined by;

$$\sqrt{M} \oplus \sqrt{N} = \sqrt{M + N}$$

This is obviously not addition in the conventional sense but when two square roots are added the result is not usually a square root. Therefore the result would not have a continued fraction of the form (a,b) . This would eliminate

the property of closure as well as introducing other complications. To avoid these difficulties this definition of addition has been chosen.

Theorem 7 : For any (a,b) and (c,d) , $(a,b) \oplus (c,d)$ is equal to $(a,b + d + c^2)$.

$$\begin{aligned}(a,b) \oplus (c,d) &= \sqrt{a^2 + b} \oplus \sqrt{c^2 + d} = \sqrt{(a^2 + b) + (c^2 + d)} \\ &= \sqrt{a^2 + (b + d + c^2)} = (a,b + d + c^2).\end{aligned}$$

Definition : The multiplication \otimes of the square roots of two natural numbers M and N is defined by;

$$\sqrt{M} \otimes \sqrt{N} = \sqrt{M \times N}$$

Theorem 8 : For any (a,b) and (c,d) , $(a,b) \otimes (c,d)$ is equal to $(ac, bc^2 + da^2 + bd)$.

$$\begin{aligned}(a,b) \otimes (c,d) &= \sqrt{a^2 + b} \otimes \sqrt{c^2 + d} = \sqrt{(a^2 + b) \times (c^2 + d)} \\ &= \sqrt{a^2c^2 + bc^2 + a^2d + bd} = (ac, bc^2 + da^2 + bd).\end{aligned}$$

The operations for this system, addition and multiplication, display many of the properties of the respective conventional operations. These properties are illustrated in the following section.

Closure:

Addition: $(a,b) \oplus (c,d) = (g,h)$ where g and h are natural numbers.

$$\begin{aligned}(a,b) \oplus (c,d) &= (a,b + d + c^2) = (g,h) \text{ where } g = a \\ &\text{and } h = b + d + c^2.\end{aligned}$$

Multiplication: $(a,b) \otimes (c,d) = (g,h)$

$$(a,b) \otimes (c,d) = (ac, bc^2 + da^2 + bd) = (g,h) \text{ where}$$

$$g = ac \text{ and } h = bc^2 + da^2 + bd.$$

Well defined:

Addition: If $(a,b) = (c,d)$ then $(a,b) \oplus (e,f) = (c,d) \oplus (e,f)$.

$$(a,b) \oplus (e,f) = (a, b + f + e^2) = \sqrt{a^2 + b + f + e^2} = \sqrt{a^2 + b + e^2 + f}$$

if $(a,b) = (c,d)$ then $a^2 + b = c^2 + d$ so by substitution;

$$\sqrt{a^2 + b + e^2 + f} = \sqrt{c^2 + d + e^2 + f} = (c,d) \oplus (e,f).$$

Multiplication: If $(a,b) = (c,d)$ then $(a,b) \otimes (e,f) =$

$$(c,d) \otimes (e,f).$$

$$(a,b) \otimes (e,f) = (ae, be^2 + fa^2 + bf)$$

$$\sqrt{a^2e^2 + bc^2 + fa^2 + bf} = \sqrt{(a^2 + b) \times (e^2 + f)}$$

if $(a,b) = (c,d)$ then $a^2 + b = c^2 + d$ so by substitution

$$\sqrt{(a^2 + b) \times (e^2 + f)} = \sqrt{(c^2 + d) \times (e^2 + f)} = (c,d) \otimes (e,f)$$

Cancellation :

Addition: If $(a,b) \oplus (e,f) = (c,d) \oplus (e,f)$ then

$$(a,b) = (c,d).$$

if;

$$(a,b) \oplus (e,f) = (c,d) \oplus (e,f)$$

then;

$$(a, b + f + e^2) = (c, d + f + e^2)$$

which implies that;

$$a^2 + b + f + e^2 = c^2 + d + f + e^2$$

or

$$a^2 + b = c^2 + d \text{ which implies that } (a,b) = (c,d)$$

Multiplication: If $(a,b) \otimes (e,f) = (c,d) \otimes (e,f)$ then
 $(a,b) \otimes (c,d)$.

if,

$$(a,b) \otimes (e,f) = (c,d) \otimes (e,f)$$

then,

$$(ae, be^2 + fa^2 + bf) = (cd, de^2 + fc^2 + df)$$

which implies that,

$$a^2e^2 + be^2 + fa^2 + bf = c^2e^2 + de^2 + fc^2 + df$$

$$(a^2 + b) \times (e^2 + f) = (c^2 + d) \times (e^2 + f)$$

or, $a^2 + b = c^2 + d$ which implies that $(a,b) = (c,d)$

Commutative :

Addition: $(a,b) \oplus (c,d) = (c,d) \oplus (a,b)$

$$\begin{aligned} (a,b) \oplus (c,d) &= (a, b + d + c^2) = \sqrt{a^2 + b + d + c^2} \\ &= \sqrt{c^2 + d + b + a^2} = (c, d + b + a^2) = (c,d) \oplus (a,b) \end{aligned}$$

Multiplication: $(a,b) \otimes (c,d) = (c,d) \otimes (a,b)$

$$\begin{aligned} (a,b) \otimes (c,d) &= (ac, be^2 + da^2 + bd) \\ &= (ca, da^2 + bc^2 + db) = (c,d) \otimes (a,b) \end{aligned}$$

Associative :

Addition: $[(a,b) \oplus (c,d)] \oplus (e,f) = (a,b) \oplus [(c,d) \oplus (e,f)]$

$$\begin{aligned} [(a,b) \oplus (c,d)] \oplus (e,f) &= (a, b + d + c^2) \oplus (e,f) \\ &= [a, (b + d + c^2) + f + e^2] = [a, b + (d + f + e^2) + c^2] = \\ &= (a,b) \oplus (c, b + f + e^2) = (a,b) \oplus [(c,d) \oplus (e,f)] \end{aligned}$$

$$\begin{aligned}
\text{Multiplication: } [(a,b) \otimes (c,d)] \otimes (e,f) &= (a,b) \otimes [(c,d) \otimes (e,f)] \\
[(a,b) \otimes (c,d)] \otimes (e,f) &= (ac, bc^2 + da^2 + bd) \otimes (e,f) \\
&= [(ac)e, (bc^2 + da^2 + bd)e^2 + f(ac)^2 + (bc^2 + da^2 + bd)f] \\
&= [ace, bc^2e^2 + da^2e^2 + bde^2 + fa^2c^2 + bc^2f + da^2f + bdf] \\
&= (ace, bc^2e^2 + da^2e^2 + bde^2 + fa^2c^2 + bde^2 + bfc^2 + bdf) \\
&= [a(ce), b(ce)^2 + (de^2 + fc^2 + df)a^2 + b(de^2 + fc^2 + df)] \\
&= (a,b) \otimes (ce, de^2 + fc^2 + df) = (a,b) \otimes [(c,d) \otimes (e,f)]
\end{aligned}$$

Distributive : Multiplication is distributive with respect to addition.

$$\begin{aligned}
(a,b) \otimes [(c,d) \oplus (e,f)] &= [(a,b) \otimes (c,d)] \oplus [(a,b) \otimes (e,f)] \\
(a,b) \otimes [(c,d) \oplus (e,f)] &= (a,b) \otimes (c, b + f + e^2) \\
&= [ac, bc^2 + (d + f + e^2)a^2 + b(d + f + e^2)] \\
&= (ac, bc^2 + da^2 + fa^2 + e^2a^2 + bd + bf + be^2) \\
&= [ac, (bc^2 + da^2 + bd) + (be^2 + fa^2 + bf) + (ae)^2] \\
&= (ac, bc^2 + da^2 + bd) \oplus (ae, be^2 + fa^2 + bf) \\
&= [(a,b) \otimes (c,d)] \oplus [(a,b) \otimes (e,f)]
\end{aligned}$$

CHAPTER III

A METHOD FOR OBTAINING SIMPLE CONTINUED FRACTIONS

The special form (a,b) obviously does not produce a simple continued fraction every time. However, this form can be used to develop a method of obtaining simple continued fraction representations of the square roots of natural numbers.

This method is based on theorem 6 and the fact that in a general continued fraction;

$$a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}}$$

the value is unaltered if b_n , a_n , and b_{n+1} are each multiplied by the same constant. Theorem 6 indicates that (a,b) can be written in more than one form. To be exact, if (a,b) is in lowest terms, it can be written in "a" different forms. These forms are;

$$(a,b) = (a-1, b + 2a - 1) = (a-2, b + 4a - 4) = \dots$$

$$= [1, b + 2a(a-1) - (a-1)^2] \quad (3-1)$$

Inspection of the continued fraction;

$$(a,b) = a + \frac{b}{2a} + \frac{b}{2a} + \frac{b}{2a} + \dots$$

shows that each partial quotient x_n has the value $a + (a,b)$. If another form of (a,b) is substituted for (a,b) it will not change the value of the continued fraction. Furthermore if this second form, say (a',b') , is substituted to form

$$(a,b) = a + \frac{b}{a + a'} + \frac{b'}{2a'} + \frac{b'}{2a'} + \dots \quad (3-2)$$

and if b divides both $(a + a')$ and b' then b can be divided out and the first step in obtaining a simple continued fraction will be completed. Repetition of this process yields the desired result.

An example is given to illustrate this method.

$$19 = (4,3) = (3,10) = (2,15) = (1,18)$$

$$\begin{aligned} (4,3) &= 4 + \frac{3}{8} + \frac{3}{8} + \frac{3}{8} + \dots \\ &= 4 + \frac{3}{4+2} + \frac{15}{4} + \frac{15}{4} + \frac{15}{4} + \dots \\ &= 4 + \frac{1}{2} + \frac{5}{4} + \frac{15}{4} + \frac{15}{4} + \dots \\ &= 4 + \frac{1}{2} + \frac{5}{2+3} + \frac{10}{6} + \frac{10}{6} + \dots \\ &= 4 + \frac{1}{2} + \frac{1}{1} + \frac{2}{6} + \frac{10}{6} + \frac{10}{6} + \dots \\ &= 4 + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{5}{6} + \frac{10}{6} + \frac{10}{6} + \dots \\ &= 4 + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{5}{3+2} + \frac{15}{4} + \frac{15}{4} + \dots \\ &= 4 + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{3}{4} + \frac{15}{4} + \frac{15}{4} + \dots \end{aligned}$$

$$\begin{aligned}
 &= 4 + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{3}{2+4} + \frac{3}{8} + \frac{3}{8} + \dots \\
 &= 4 + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{2} + \frac{1}{8} + \frac{3}{8} + \dots
 \end{aligned}$$

Now the partial quotient beginning with $\frac{3}{8}$ is the same as

the original and the process repeats again. It will be noticed that in the fifth step no substitution was required to reduce the numerator to one. In many continued fractions this will be the case, thereby shortening the process considerably. Also it will be noticed that in the ninth step either (4,3) or (1,18) could be substituted for (a,b). On page 11 of the first chapter in the development of a simple continued fraction $a_1 = [p_1]$. If this method is to be valid (a',b') must be chosen so that a' has the largest possible value.

Two main questions must be answered before this method can be put into general use; (1) Can the continued fraction (a',b') which is equal to (a,b) be found such that b divides (a + a')?, (2) If such a continued fraction is found will b also divide b'? These questions are answered in the two following theorems.

Theorem 9 : If (a,b) is in lowest terms and b is not zero then (a',b') equal (a,b) can be found such that b divides (a + a').

If (a,b) is in lowest terms and $b \neq 0$ then by theorem 5, $0 < b \leq 2a$. From theorem 6, a' could take values

$(a, a-1, a-2, \dots, 1)$, therefore $a + a'$ could take values $(a + a, a + a - 1, \dots, a + 1)$ or a consecutive numbers from $a + 1$ to $2a$ inclusive. If b is between 1 and a inclusive there exists at least one number in any a consecutive numbers that is divisible by b . If b is between $a + 1$ and $2a$ inclusive the values which $(a + a')$ may take include the same set of numbers.

Therefore for $0 < b \leq 2a$ there exist some number $a < a + a' \leq 2a$ such that b divides $a + a'$.

Theorem 10 : If $(a, b) = (a', b')$ and b divides $a + a'$ then b also divides b' .

If $(a, b) = (a', b')$ then $a^2 + b = (a')^2 + b'$

and $a^2 - (a')^2 + b = b'$

$$(a + a')(a - a') + b = b'$$

but;

b divides $a + a'$ and b divides b ,

therefore b also divides b' .

CONCLUSION

An algebraic system for continued fractions which represent the positive square roots of natural numbers has been developed. These continued fractions, which could be represented by an ordered pair notation (a,b) , were shown to follow a definite pattern. Using this notation equality was defined and was shown to be a true equivalence relationship. After defining the operations addition and multiplication it was shown that these operations follow the commutative and associative laws and that the distributive law of multiplication with respect to addition was valid.

In connection with the work on continued fractions having the form (a,b) , many interesting questions arose which were not completely answered in this paper. One possibility that poses several questions is to allow a and b to take all possible integral values. When a takes the value zero, the continued fraction takes a degenerate form so $(0,b)$ could not be allowed in the system. Other forms which could be considered in the system are $(a,-b)$, $(-a,b)$, and $(-a,-b)$. With these additions a and b could retain their original ranges while the members of the ordered pairs could take all possible forms. When evaluated these forms have the values;

$$(a,b) = \sqrt{a^2 - b} , (-a,b) = -\sqrt{a^2 + b} , (-a,-b) = -\sqrt{a^2 - b} .$$

This immediately leads to the question, 'Do $(a, -b)$ or $(-a, -b)$ produce imaginary numbers if b is larger than a^2 ?'. This question, whose answer is surely no, cannot be completely answered without further investigation. However in the specific cases which have been investigated the successive convergents either diverge or oscillate thereby making the infinite continued fraction representation invalid.

If these three forms, with the proper conditions placed on a and b , were incorporated into the system, the operation of subtraction could be defined and both an additive identity $(1, -1)$ and a multiplicative identity $(1, 0)$ could be introduced.

A method for obtaining a simple continued fraction representation for the positive square root of a natural number has been developed. This method is a step by step process in which one basic step is repeated until the simple continued fraction is obtained. It was proven that the first step could always be carried out but it was not rigorously proven that the step could always be repeated. This could not be accomplished without a more extensive study of the (a, b) form of continued fraction. From the development of simple continued fractions and the fact that one and only one simple continued fraction exists which represents \sqrt{N} (where N is a natural number and not a perfect square) it

would follow intuitively that the repetition of this step is always possible.

The simple continued fraction representing \sqrt{N} was shown in chapter one to have a special form. The general quadratic irrational $\frac{a + b\sqrt{N}}{c}$ where a , b , c , and N are all natural numbers (a could be zero also) can also be represented as a simple continued fraction. The extension of the method contained in this paper to include those simple continued fractions which represent $\frac{a + b\sqrt{N}}{c}$ would make an interesting project.

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